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Spanning local tournaments in locally semicomplete digraphs ☆

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Abstract

We investigate the existence of a spanning local tournament with possibly high connectivity in a highly connected locally semicomplete digraph. It is shown that every $(3k-2)$-connected locally semicomplete digraph contains a k-connected spanning local tournament. This improves the result of Bang-Jensen and Thomassen for semicomplete digraphs and of Bang-Jensen [1] for locally semicomplete digraphs.

Keywords: Digraph; Connectivity

1. Terminology and preliminaries

We denote by $V(D)$ and $E(D)$ the vertex set and the arc set of a digraph D, respectively. If xy is an arc of D, then we say that x *dominates* y. More generally, if A and B are two disjoint subdigraphs of D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B, denoted by $A \rightarrow B$. In addition, if $A \rightarrow B$, but there is no arc from B to A, then we say that A *strictly dominates* B, denoted by $A \Rightarrow B$.

The *outset* of a vertex $x \in V(D)$ is the set $N^+(x) = \{y \mid xy \in E(D)\}$. Similarly, $N^-(x) = \{y \mid yx \in E(D)\}$ is the *inset* of x. More generally, for a subdigraph A of D, we define its *outset* by $N^+(A) = \bigcup_{x \in V(A)} N^+(x) - A$ and its *inset* by $N^-(A) = \bigcup_{x \in V(A)} N^-(x) - A$ (if necessary, we write $N_D^+(A)$ and $N_D^-(A)$ instead of $N^+(A)$ and $N^-(A)$, respectively). Every vertex of $N^+(A)$ is called an *out-neighbour* of A and every vertex of $N^-(A)$ is an *in-neighbour* of A.

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In this paper we shall consider finite digraphs without loops and multiple arcs. The numbers $d^+(x)=|N^+(x)|$ and $d^-(x)=|N^-(x)|$ are called *outdegree* and *indegree* of $x \in V(D)$, respectively (if necessary, we write $d_D^+(x)$ and $d_D^-(x)$ instead of $d^+(x)$ and $d^-(x)$, respectively). If $d^+(x)=d^-(x)=r$ holds for every vertex x of D, then we say that D is r-regular.

Paths and cycles in a digraph are always directed. A path from x to y is called an (x,y)-path. Two (x,y)-paths are *internally disjoint* if they have only the vertices x and y in common.

The subdigraph of D induced by a subset A of $V(D)$ is denoted by $D\langle A\rangle$. In addition, $D-A=D\langle V(D)-A\rangle$.

A *strong component* H of D is a maximal subdigraph such that for any two vertices $x,y \in V(H)$, the subdigraph H contains a path from x to y and a path from y to x. The digraph D is *strong* or *strongly connected*, if it has only one strong component, and D is k-connected if for any set A of at most $k-1$ vertices, the subdigraph $D-A$ is strong.

If D is strong and S is a subset of $V(D)$ such that $D-S$ is not strong, then we say that S is a *separating set* of D. A separating set S of D is *minimal* if for any proper subset S' of S, the subdigraph $D-S'$ is strong.

A digraph is *connected*, if its underlying graph is connected. In this paper, we only consider connected digraphs.

If we replace every arc xy of D by yx, then we call the resulting digraph (denoted by D^{-1}) the *converse digraph* of D.

A subdigraph of D is called a *spanning subdigraph* if it contains all vertices of D.

A digraph is *semicomplete* if for any two different vertices x and y, there is at least one arc between them. A *tournament* is a semicomplete digraph without a cycle of length two.

In 1990, Bang-Jensen [1] introduced a very interesting generalization of tournaments – the class of locally semicomplete digraphs. A digraph D is *locally semicomplete*, if for every vertex x, $D\langle N^+(x)\rangle$ and $D\langle N^-(x)\rangle$ are both semicomplete. A *local tournament* is a locally semicomplete digraph without a cycle of length two. It is obvious that the class of locally semicomplete digraphs is a superclass of local tournaments.

We note that every induced subdigraph of a locally semicomplete digraph is locally semicomplete. In addition, the converse of a locally semicomplete digraph is also locally semicomplete.

It has been shown that locally semicomplete digraphs have many properties in common with tournaments. The first of them is the following:

Theorem 1.1 (Bang-Jensen [1]). *Every strong locally semicomplete digraph contains a hamiltonian cycle.*

In other words, every strong locally semicomplete digraph contains a spanning local tournament that is also strong.

In general, the existence of a spanning local tournament with possibly high connectivity in a highly connected locally semicomplete digraph is an interesting problem.

Bang-Jensen and Thomassen (cf. [1]) proved that every $5k$-connected semicomplete digraph contains a k-connected spanning tournament. Bang-Jensen [1] generalized this result to locally semicomplete digraphs and proved that every $5k$-connected locally semicomplete digraph contains a k-connected spanning local tournament. However, he conjectured the following:

Conjecture 1.2 (Bang-Jensen [1]). Every $2k$-connected locally semicomplete digraph contains a k-connected spanning local tournament.

In this paper we prove that a $(3k-2)$-connected locally semicomplete digraph contains a k-connected spanning local tournament (see Theorem 2.2). So, Conjecture 1.2 is also true for $k=2$.

To prove our main results we need the following results.

Theorem 1.3 (Bang-Jensen [1]). *Let D be a connected locally semicomplete digraph that is not strong. Then the following holds*:

(a) *If A and B are two strong components of D, then either there is no arc between them or $A \Rightarrow B$ or $B \Rightarrow A$.*

(b) *If A and B are two strong components of D such that A dominates B, then A and B are both semicomplete digraphs.*

(c) *The strong components of D can be ordered in a unique way $D_1, D_2, \ldots, D_p$ such that there are no arcs from D_j to D_i for $j > i$, and D_i dominates D_{i+1} for $i = 1, 2, \ldots, p-1$.*

The unique sequence $D_1, D_2, \ldots, D_p$ of the strong components of D in Theorem 1.3 (c) is called the *strong decomposition* of D with the *initial* component D_1 and the *terminal* component D_p.

Theorem 1.4 (Guo and Volkmann [3]). *Let D be a connected locally semicomplete digraph that is not strong and let $D_1, \ldots, D_p$ be the strong decomposition of D. Then D can be decomposed in $r \geqslant 2$ subdigraphs $D'_1, D'_2, \ldots, D'_r$ as follows*:

$$D'_1 = D_p, \qquad \lambda_1 = p,$$

$$\lambda_{i+1} = \min\{j \mid N^+(D_j) \cap V(D'_i) \neq \emptyset\}$$

and

$$D'_{i+1} = D\langle V(D_{\lambda_{i+1}}) \cup V(D_{\lambda_{i+1}+1}) \cup \cdots \cup V(D_{\lambda_i - 1})\rangle.$$

Furthermore, the subdigraphs $D'_1, D'_2, \ldots, D'_r$ satisfy the following:

(a) *D'_i consists of some strong components of D and it is semicomplete for $i = 1, 2, \ldots, r$;*

(b) D'_{i+1} *dominates the initial component of* D'_i *and there exists no arc from* D'_i *to* D'_{i+1} *for* $i=1,2,\ldots,r-1$;
(c) *if* $r \geqslant 3$, *then there is no arc between* D'_i *and* D'_j *for* i,j *satisfying* $|j-i| \geqslant 2$.

For a connected, but not strongly connected locally semicomplete digraph D, the unique sequence $D'_1, D'_2, \ldots, D'_r$ defined in Theorem 1.4 is called the *semicomplete decomposition* of D.

Lemma 1.5 (Bang-Jensen [1]). *Let D be a strong locally semicomplete digraph and let S be a minimal separating set of D. Then $D-S$ is connected.*

Lemma 1.6. *Let D be a k-connected digraph and let A be a vertex set with $A \cap V(D) = \emptyset$. If we add some arcs between A and D such that* $\min\{d^+(v), d^-(v)\} \geqslant k+1$ *for every $v \in A$, then the resulting digraph H is also k-connected. Moreover, if H is not $(k+1)$-connected, then every minimum separating set of H is also a minimum separating set of D.*

Proof. Suppose that H is not $(k+1)$-connected. Let S be a minimum separating set of H. Then $|S| \leqslant k$. Because of $d^+(v), d^-(v) \geqslant k+1$ for each $v \in A$, every vertex of $A-S$ has at least one out-neighbour and at least one in-neighbour in the subdigraph $D-S$. It follows that $D-S$ is not strong. Since D is k-connected, we see that $|S|=k$. Therefore, H is k-connected and every minimum separating set of H is also a minimum separating set of D. □

Lemma 1.7. *Let D be a k-connected digraph and $xy \in E(D)$. If D contains at least $k+1$ internally disjoint (x,y)-paths, each of which is of length at least 2, then the digraph D' obtained from D by replacing xy with yx is also k-connected. Furthermore, if D' is not $(k+1)$-connected, then every minimum separating set of D' is also a separating set of D.*

Proof. Suppose that D' is not $(k+1)$-connected. Let S be a minimum separating set of D' with $|S| \leqslant k$. Then there are two vertices a and b such that $D'-S$ contains no (a,b)-path. If $S \cap \{x,y\} \neq \emptyset$, then it is obvious that $D-S$ also contains no (a,b)-path. Thus $D-S$ is not strong. If $S \cap \{x,y\} = \emptyset$, then it is easy to see that x and y are in the same strong component of $D'-S$, and hence $D-S$ contains no (a,b)-path. This means that $D-S$ is not strong. Therefore, every minimum separating set of D' is also a separating set of D. Since D is k-connected, $|S|=k$ holds. Thus D' is k-connected. □

2. Main results

We first consider the existence of a spanning local tournament with possibly high connectivity in a highly connected locally semicomplete digraph which is not semicomplete.

Theorem 2.1. *Let D be a $(2k-1)$-connected locally semicomplete digraph which is not semicomplete. Then D contains a k-connected spanning local tournament.*

Proof. We shall show the statement by induction on the number k. If $k=1$, then, by Theorem 1.1, D has a Hamiltonian cycle and we are done. So we may assume that $k \geqslant 2$. Since D is not semicomplete, D has a minimal separating set S such that $D-S$ is not semicomplete. According to Lemma 1.5, $D-S$ is connected. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition and let $D_1', D_2', \ldots, D_r'$ be the semicomplete decomposition of $D-S$, respectively. Obviously, $r \geqslant 3$ and $|S|, |V(D_2')| \geqslant 2k-1 \geqslant 3$. Using the definition of locally semicomplete digraphs and Theorem 1.4, it is not difficult to show that $D_2' \Rightarrow D_1' \Rightarrow S \Rightarrow D_1$.

Let $x_1 \in V(D_1)$, $x_2 \in V(D_p)$ and $D' = D - \{x_1, x_2\}$. It is clear that D' is $(2(k-1)-1)$-connected. By the induction hypothesis, D' contains a $(k-1)$-connected spanning local tournament T'. Now we consider the subdigraph D'' of D obtained from T' by adding the two vertices x_1, x_2 and all the arcs between T' and $\{x_1, x_2\}$. From the choice of the two vertices x_1, x_2 and the assumption that D is $(2k-1)$-connected, we can destroy in D'' all cycles of length 2 between T' and $\{x_1, x_2\}$ such that every vertex of $\{x_1, x_2\}$ has at least k out-neighbours and k in-neighbours in T'. Thus, we obtain a spanning local tournament T of D. By Lemma 1.6, T is $(k-1)$-connected.

Suppose that T is not k-connected. Then T has a separating set S' with $|S'| = k-1$. By Lemma 1.6, S' is also a separating set of T'. It follows that x_1 and x_2 do not belong to S'. Let $T_1, T_2, \ldots, T_q$ be the strong decomposition and let $T_1', T_2', \ldots, T_t'$ be the semicomplete decomposition of $T-S'$, respectively. Since x_1 and x_2 are not adjacent, $T-S'$ is not semicomplete. It follows that $t \geqslant 3$. Thus, S' is the outset of $V(T_q)$ in T. Because T is a spanning local tournament of D, S' is also the outset of $V(T_q)$ in D. This contradicts the assumption that D is $(2k-1)$-connected. Therefore, T is k-connected. □

In the proof of the next theorem, we will use some ideas of Bang-Jensen (see the proof of Theorem 4.5 in [1]).

Theorem 2.2. *A $(3k-2)$-connected locally semicomplete digraph contains a k-connected spanning local tournament.*

Proof. Let D be a $(3k-2)$-connected locally semicomplete digraph on n vertices. If D is not semicomplete, then we are done by Theorem 2.1, since $3k-2 \geqslant 2k-1$ for all $k \geqslant 1$. So we assume that D is semicomplete. We shall prove the statement by induction on the number k.

If $k=1$, then D has a Hamiltonian cycle by Theorem 1.1. Assume thus $k \geqslant 2$. If D is complete, then it is a simple matter that D contains a k-connected spanning tournament. So we only need consider the case that D is not complete. This implies that $n \geqslant 3k$. Because of $3k-2 > 3(k-1)-2$, D contains a $(k-1)$-connected spanning tournament by the induction hypothesis.

Suppose that D does not contain a k-connected spanning tournament. Let T be chosen among all $(k-1)$-connected spanning tournaments of D such that the following holds:

(1) The number of minimum separating sets of T is as small as possible.

(2) T has a minimum separating set S such that the number of strong components of $T-S$ is as small as possible.

Let $T_1, T_2, \dots, T_p$ be the strong components of $T-S$. Suppose without loss of generality that $|V(T_1)| \geqslant |V(T_p)|$ (otherwise, we consider the converse D^{-1} of D). Clearly, the semicomplete decomposition of $T-S$ has exactly two components T_1' and T_2' and

$$2|V(T_2')| \geqslant |V(T_1')| + |V(T_2')| = n - |S| \geqslant 3k - (k-1) = 2k+1.$$

It follows that $|V(T_2')| \geqslant k+1$. Let $A_1 = N_D^-(T_2') \cap V(T_1')$ and $A_2 = N_D^+(T_1') \cap V(T_2')$. Since D is $(3k-2)$-connected, $|A_i| \geqslant \min\{2k-1, |V(T_i')|\}$ holds for $i=1,2$. We shall show that there are two vertices $x \in V(T_\mu)$, $y \in V(T_\nu)$ satisfying the following conditions:

(a) $1 \leqslant \mu < \nu \leqslant p$ and $|V(T_\mu) \cup \cdots \cup V(T_\nu)| \geqslant 3$.

(b) yx is an arc of D.

(c) T contains at least k internally disjoint (x,y)-paths each of length at least 2.

Suppose $|V(T_1')| \geqslant 2k$. Let A_1' be a subset of A_1 with $|A_1'| = 2k-1$. Since T_1' is strong, there is at least one arc from $T_1' - A_1'$ to A_1'. So we have

$$\sum_{v \in A_1'} d_{T_1'}^-(v) = \frac{(2k-1)(2k-2)}{2} + 1.$$

It follows that A_1' contains a vertex y having at least k in-neighbours $y_1, y_2, \dots, y_k \in V(T_1')$. Let x be a vertex of A_2 such that yx is an arc of D. Because of $x \Rightarrow T_1'$, T contains k internally disjoint (x,y)-paths $x \to y_i \to y$ for $i = 1, 2, \dots, k$.

Suppose now that $|V(T_1')| = \alpha \leqslant 2k-1$. It follows that $A_1 = V(T_1')$.

Assume that T_1' contains a vertex y which has at least $\lfloor \alpha/2 \rfloor + 1$ in-neighbours in T_1'. Let $A_2' = N_D^+(y) \cap A_2$. Since D is $(3k-2)$-connected, $|A_2'| \geqslant (3k-2) - |S| - (\alpha - 1) = 2k - \alpha$. It is easy to see that A_2' contains a vertex x having at least

$$\left\lceil \frac{(2k-\alpha)(2k-\alpha-1)}{2} \cdot \frac{1}{2k-\alpha} \right\rceil = k - \left\lfloor \frac{\alpha-1}{2} \right\rfloor$$

out-neighbours in T_2'. Because of $k - \lfloor(\alpha-1)/2\rfloor + (\lfloor \alpha/2 \rfloor + 1) \geqslant k$, we see that T has at least k internally disjoint (x,y)-paths of length 2.

Assume thus that every vertex of T_1' has at most $\lfloor \alpha/2 \rfloor$ in-neighbours in T_1'. This implies that T_1' is regular, and hence the number $\alpha = |V(T_1')|$ is odd. Let $\alpha = 2t + 1$ with $t \geqslant 0$. Furthermore, we may assume that every vertex of A_2 has at most $k-t-1$ out-neighbours in T_2'. By the observation $|A_2| \geqslant 2k - \alpha$, it is easy to check that $A_2 = V(T_{p-1})$, $|A_2| = 2k - \alpha$ and T_{p-1} is also regular. Now we note that there are exactly $k-1$ internally disjoint (x,y)-paths in $T\langle V(T_{p-1}) \cup V(T_p)\rangle$ for any two vertices $x \in V(T_{p-1})$ and $y \in V(T_p)$.

Since $n \geqslant 3k$ and $|V(T_p)| + |V(T_{p-1})| = 2k$, we have $p \geqslant 3$. Suppose that T contains an arc xs from T_{p-1} to S. Because of $s \to z \to T_p$ for some $z \in V(T_1)$, T contains k

internally disjoint (x, y)-paths for any $y \in V(T_p)$. So we may assume that $S \rightarrow T_{p-1}$. Since D is $(3k - 2)$-connected and $A_2 = V(T_{p-1})$, $|N_D^-(T_2' - V(T_{p-1}))| \geqslant 3k - 2$. It follows that $\alpha = 1$, and hence T_{p-1} is a $(k - 1)$-regular tournament. Thus $N_D^-(T_2' - V(T_{p-1})) = S \cup V(T_{p-1})$. Let x be a vertex of $T_2' - V(T_{p-1})$ and $y \in V(T_{p-1})$ such that yx is an arc of D. Then we see that the $k - 1$ in-neighbours of y in T_{p-1} and the path $x \rightarrow T_p \rightarrow s \rightarrow y$ with $s \in S$ form k internally disjoint (x, y)-paths.

Altogether, we have shown that $T - S$ contains two vertices x and y satisfying the conditions (a)–(c) above.

Now we consider the tournament T' obtained from T by replacing the arc xy with yx. From the assumption that D contains no spanning k-connected tournament, we see that T' is not k-connected. By Lemma 1.7, T' is $(k-1)$-connected and every minimum separating set of T' is also one of T.

If $T' - S$ is strong, then the number of minimum separating sets of T' is less than that of T, a contradiction to the choice of T in view of the condition (1) above. So we assume that $T' - S$ is not strong. Since $T'\langle V(T_\mu) \cup \cdots \cup V(T_\nu)\rangle$ is strong, the number of the strong components of $T' - S$ is less than that of $T - S$. This contradicts the conditions (1) and (2).

Therefore, D contains a k-connected spanning tournament. □

Corollary 2.3 (Bang-Jensen [1]). *A 5k-connected locally semicomplete digraph has a k-connected spanning local tournament.*

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